Localisation and Nonlocality in Compound Energy-Momentum Eigenstates

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A thought experiment considering conservation of energy and momentum for a pair of free bodies together with their internal energy is used to show the existence of states that have localised position while being eigenstates of energy and momentum. These states are applicable to all varieties of physical bodies, including planets and stars in free motion in the universe. The states are compound entanglements of multiple free bodies in which the momenta of the bodies are anticorrelated so that they always sum to zero, while their total kinetic energy is anticorrelated with their internal energies, so the total is a constant, E. The bodies are relatively localised while the total state has well-defined energy and momentum. These states do not violate Heisenberg uncertainty because the total centre of mass is not localised, hence the states naturally describe whole universes rather than isolated systems within a universe. A further property of these states, resulting from the form of the entanglement, is that they display nonlocality in the full sense of signal transmission rather than the more restricted Bell sense.
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0 Introduction

One of the fundamental results of quantum theory is the Heisenberg uncertainty principle. If two observables are represented by non-commuting operators then it is not possible for those observables to simultaneously have precise values. The first example normally mentioned is position-momentum uncertainty: the more precisely position is known, the less precisely momentum will be known,

and vice-versa. This leads to the expectation that a body in free space cannot simultaneously occupy a momentum eigenstate and be localised, that is, have its position defined to within narrow limits.

The purpose of the present work is to show that this expectation, while strictly correct, is misleading in an important way. It is possible for bodies to be localised and yet at the same time to occupy a momentum eigenstate. The smallest state permitting this requires two bodies whose momenta are anticorrelated, that is, are precisely equal and opposite. In this case the two bodies are relatively localised even though jointly occupying an eigenstate of their total momentum having eigenvalue 0. The centre of mass state is completely unlocalised, as demanded by the uncertainty relations. This state is essentially the one described by Einstein, Podolsky and Rosen in their famous paper.[2]

Because the centre of mass is not localised, the above state would appear to be of no practical importance beyond specialised uses: the bodies are localised relative to each other, but are not localised relative to the rest of the universe. The present paper shows, however, that a state can be constructed containing any arbitrary number of bodies, with all bodies localised relative to each other, and with total momentum 0. If such a state described the entire universe, the lack of absolute centre of mass localisation would be irrelevant as there would be no external observer, and observers within the universe would find all observable bodies localised.

The paper also shows that the two-body state can be extended in a different way, so that the total kinetic energy of the bodies is placed in correlation with the internal energy of the bodies so that their sum is a constant value E, thus creating an eigenstate of total energy. This can be done while retaining the relative localisation and precisely defined total momentum, opening the possibility that the universe as a whole occupies an energy-momentum eigenstate. It is not proved, however, that this can be extended to arbitrarily large numbers of bodies, although it appears highly unlikely that this extension would fail.

Localised energy-momentum eigenstates have a further pair of remarkable properties. The first is that they violate the well-known dictum that interference cannot continue if we are in a position to determine which of the interfering states the system occupies. Localisation is a phenomenon resulting from interference between free-body momentum states, but these states are now in correlation with the internal energy states of the bodies. If the internal energy were measured the momentum state of the system would be known, yet the interference remains.

The second property derives from the first. If we actually measure the internal energy of the bodies each of them will be placed in a momentum eigenstate, a state that is completely unlocalised. This loss of localisation would instantly affect the whole universe, and would be readily observable, allowing it to be used as a means of superluminal signal transmission. These localised energy-momentum eigenstates are therefore also *nonlocal*, in the entirely dif-

ferent sense of permitting physical influences to travel across space. Moreover, unlike the Bell nonlocality,[4] these influences appear to be usable to transmit signals, and so may profoundly alter our present ideas about the relationship between quantum theory and special relativity.

1 A Thought Experiment

Consider a solid body. Make the normal separation between the centre of mass state and the internal state. Initially, the internal state is an excited energy eigenstate of energy E at least sufficient to eject one electron from the body. Such states are also momentum eigenstates of eigenvalue 0. At a later time one electron has been emitted from this body. (It might be useful to picture the body as an electron gun before and after emitting.) The state is now composed of three parts: the internal state of the body, lacking one electron; the state describing the relative position of the body and the electron; and the state of the centre of mass of the total system. The first two of these are not necessarily separable from each other, but they are separable from the third. We can reasonably expect the state of the body plus electron to have the following properties.

i. The electron and the body are localised relative to each other.

ii. The total momentum of the body and electron in the centre of mass frame is described by a momentum eigenstate of momentum 0.

iii. The relative kinetic energy of the body and electron, plus the internal energy of the body, is described by an energy eigenstate of energy E.

iv. The centre of mass state of the total body plus electron system is the same as the original centre of mass state of the body.

Requirement i is demanded by our normal empirical observation that particles emitted in situations like this are localised. Requirements ii to iv are demanded by the relevant conservation laws, together with the technique of separation of internal and centre of mass states for systems.

It may appear at first sight that the condition i is incompatible with conditions ii and iii: the Heisenberg uncertainty relations between position and momentum appear to forbid i and ii from being realised at once. The same intuition should also apply to the combination of i and iii, since the state involves free particle kinetic energy eigenstates, which essentially are the momentum eigenstates. This appearance will turn out to be misleading, as it must do. If i, ii and iii cannot be combined then either energy-momentum conservation or the known behaviour of emitted particles will be violated.

We have the standard theorem that Hamiltonian time-evolution preserves the mean value and distribution of the energy of a quantum system if

the Hamiltonian is time independent,[1] which it is presumed to be for an isolated system. This theorem assures us that ii and iii must hold, and if they fail then the status of conservation laws in quantum mechanics will be called into question.

The main argument of this paper proves that a quantum state conforming to the above requirements can exist by constructing an example of such a state. The example is not the most physically natural such state, it is chosen to make the mathematical and conceptual analysis as simple as possible. It may be helpful to briefly describe the course of the argument in prose.

I can give an intuitive feel for the form a state like this will take. To begin with I consider only the relative position and momentum requirements (i and ii). When the electron is emitted it has a momentum distribution, which for the moment I will assume to be Gaussian. Conservation of momentum demands that the body recoil with equal and opposite momentum, which amounts to a requirement that the body's momentum be anticorrelated with that of the electron. So the two momenta are anticorrelated and Gaussian distributed. The total momentum of the state is clearly zero in the centre of mass frame, so the state is an eigenstate of momentum. After transforming from the momentum basis to the position basis the relative position of the electron and body will be Gaussian distributed.

Now, this state is so far not an energy eigenstate: anticorrelation of the momenta still leaves a state of variable kinetic energy. The residual energy, the difference between the kinetic energy and the total energy E, remains in the internal state of the body. Hence the internal state cannot be an energy eigenstate, it must involve a distribution of energies anticorrelated with the kinetic energy of the body-electron state. I will be obliged to change the momentum distribution: the internal energy has a lower bound fixed by the lowest available internal energy state. This will impose an upper bound on the kinetic energy.

I will try to move toward the expected state in stages; first by combining the localisation requirement i with the momentum requirement ii. Then I will add the energy conservation requirement. Since only the form of the solutions is important, I set $\hbar = c = 1$ and set all normalisation constants to one, without any loss of generality.

2 Momentum and Localisation

In this section I find a state, $|LK\rangle$ (for localised k-eigenstate), that satisfies only the localisation and momentum conservation requirements above. It is easy to write down the state satisfying requirement ii; it can be written directly in the momentum basis as a momentum anticorrelated state. I will assume a Gaussian distribution of momentum, this being the normal choice for free particle localised

[1] Messiah[1], p195 and p210. It is surprising how few texts contain this fundamental result.

states. It is then necessary to change to the position basis to see whether the state has the desired localisation.

$$|LK\rangle = \int_{k=-\infty}^{\infty} e^{-k^2/\sigma_k^2} e^{ikx_{12}} \left|k\right\rangle_1 \otimes \left|-k\right\rangle_2 dk \tag{1}$$

Where σ_k is the standard deviation of the k-distribution and x_{12} is the mean separation of the bodies, 1 and 2, which determines a relative phase factor between k-eigenvectors. In general, integer subscripts identify which body the symbol refers to. To confirm formally that this is an eigenstate of momentum, apply

$$\mathbf{P}_{12} = \mathbf{P}_1 \otimes I_2 + I_1 \otimes \mathbf{P}_2 \tag{2}$$

where $\mathbf{P}_{12}$ is the total momentum operator, $\mathbf{P}_1$ is the momentum operator associated with body one, I_2 is the identity operator associated with body two, and so on.

Changing to the position basis gives

$$|LK\rangle = \int_{x_1} \int_{x_2} \int_{k=-\infty}^{\infty} e^{-k^2/\sigma_k^2} e^{ik(x_1 - x_2 + x_{12})} dk \left|x_1\right\rangle_1 \otimes \left|x_2\right\rangle_2 dx_1 dx_2 \tag{3}$$

(Where integration limits are not stated they are presumed to be from $-\infty$ to ∞.) This integral can be found from the standard Fourier transform of a Gaussian, hence

$$= \frac{2\sqrt{\pi}}{\sigma_x} \int_{x_1} \int_{x_2} e^{-(x_1 - x_2 + x_{12})^2/\sigma_x^2} \left|x_1\right\rangle_1 \otimes \left|x_2\right\rangle_2 dx_1 dx_2 \tag{4}$$

This is a Gaussian localised state in $x_1 - x_2$ with mean separation x_{12} and standard deviation $\sigma_x = 2/\sigma_k$. The two bodies are localised relative to each other: if (say) x_1 is fixed then x_2 will be known to within a Gaussian of mean $x_1 + x_{12}$ and standard deviation σ_x. Localisation here is only relative; since $|LE\rangle$ is an eigenstate of total momentum it cannot be expected to contain any centre of mass position information and it does not: all values of x_1 are equally probable if x_2 is unknown.

$|LK\rangle$ is a near descendant of the state described by Einstein, Podolsky and Rosen;[2] it is in fact a real world version of it. EPR's state involved perfectly anticorrelated momenta and perfectly correlated relative positions described by Dirac deltas. This latter was associated with an equal amplitude for all possible relative momenta. EPR's state is an eigenstate of total momentum, meaning it contains no centre of mass position information; and contains no relative momentum information, allowing it to be a relative position eigenstate. The new state $|LK\rangle$ remains an eigenstate of total momentum and so leaves the centre of mass position undefined, but the Gaussian distribution imposed on the single particle momenta leaves the positions correlated only up to a Gaussian.

So $|LK\rangle$ is a state that satisfies the first two requirements of momentum conservation and relative localisation. I have chosen to leave energy conservation aside for now, but the final requirement, that of preserving the centre of mass state, should apply. Here there is a problem because I have just argued that $|LK\rangle$ cannot contain centre of mass position information: $|LK\rangle$ is consistent only with a completely delocalised centre of mass. It is clear that if the centre of mass position is localised the perfect anticorrelation of momentum will be lost, so the state will no longer be a momentum eigenstate.

Briefly, to prove this, consider a state, $|LKL_{CM}\rangle$, obtained by imposing an effective centre of mass localisation on $|LK\rangle$ (and setting $x_{12} = 0$ and $\sigma_x = 1$ for simplicity)

$$\begin{aligned} |LKL_{CM}\rangle &= \int_{x_1}\int_{x_2} e^{-x_1^2} e^{-(x_1-x_2)^2} \left|x_1\right\rangle_1 \otimes \left|x_2\right\rangle_2 dx_1 dx_2 \\ &= \pi \int_{k_1}\int_{k_2} e^{-k_2^2/4} e^{-(k_1+k_2)^2/4} \left|k_1\right\rangle_1 \otimes \left|k_2\right\rangle_2 dk_1 dk_2 \qquad (5) \end{aligned}$$

(by repeated use of integrals 3.923 in Gradshteyn[3], p.520) This state is clearly not anticorrelated in momentum and is therefore not a momentum eigenstate.

States like $|LK\rangle$ may certainly exist in our universe, but they cannot describe the ordinary objects we see around us, because ordinary objects have localised centres of mass. A pair of bodies in the state $|LK\rangle$ would be unobservable to us, and it seems unlikely that bodies could remain in such a state for long. Interaction with the background electromagnetic radiation would eventually cause them to become localised relative to us, a process that should occur under any of the current competing theories of measurement. Hence $|LK\rangle$ does not solve the original problem of momentum conservation in particle emission, and cannot be used to describe the centre of mass states of normal objects. It looks as though i and iv are not consistent with each other.

There is an alternative view of the situation, however. If the universe held only two planets then $|LK\rangle$ would be a good description for them consistent with our expectations. In particular, the lack of centre of mass localisation is just what we expect if we believe that position is meaningful only relative to other objects. To a person standing on one of the two planets, the other would always be localised while the centre of mass position would be unobservable.

$|LK\rangle$ is not a useful description of bodies in our universe, but it is an acceptable description of a universe containing only two bodies. Unfortunately, therefore, it does not address the problem originally posed. The question immediately occurs of whether $|LK\rangle$ can be extended to describe universes with larger numbers of bodies, with all bodies pairwise relatively localised. In this case there would be the original two bodies plus a third where an external observer could stand. The first two bodies would be each localised relative to the external observer as well as to each other, so the centre of mass of these two would be effectively localised. The centre of mass of the total three-body sys-

tem would not be localised, but again this is no problem as there is no external object to which it could be related.

I have reached the point of proposing a single extended form of $|LK\rangle$, which I will call $|LKn\rangle$, encompassing every body in an n-body universe. It would have the two remarkable properties of being a momentum eigenstate and of having no absolute centre of mass localisation. To prove that such a state is possible, begin with the three body state, $|LK3\rangle$.

$$|LK3\rangle = \int_{k_1}\int_{k_2} e^{-k_1^2/\sigma_k^2} e^{ik_1x_{13}} e^{-k_2^2/\sigma_k^2} e^{ik_2x_{23}} \, |k_1\rangle_1 \otimes |k_2\rangle_2 \otimes |-k_1 - k_2\rangle_3 \, dk_1 dk_2 \tag{6}$$

This is a momentum eigenstate of eigenvalue 0. To check its localisation properties transform to the position basis.

$$= \int_{x_1}\int_{x_2}\int_{x_3}\int_{k_1} e^{-k_1^2/\sigma_k^2} e^{ik_1(x_1 - x_3 + x_{13})} \int_{k_2} e^{-k_2^2/\sigma_k^2} e^{ik_2(x_2 - x_3 + x_{23})} dk_2 dk_1 \times$$
$$\times \, |x_1\rangle_1 \otimes |x_2\rangle_2 \otimes |x_3\rangle_3 \, dx_1 dx_2 dx_3 \tag{7}$$

Applying the standard Fourier transform twice in succession gives

$$= \frac{4\pi}{\sigma_x^2} \int_{x_1}\int_{x_2}\int_{x_3} e^{-(x_1 - x_3 + x_{13})^2/\sigma_x^2} e^{-(x_2 - x_3 + x_{23})^2/\sigma_x^2} \times$$
$$\times \, |x_1\rangle_1 \otimes |x_2\rangle_2 \otimes |x_3\rangle_3 \, dx_1 dx_2 dx_3 \tag{8}$$

$|LK3\rangle$ is a state Gaussian localised in $x_1 - x_3$ and in $x_2 - x_3$. If any one of the variables x is known then the other two will be Gaussian localised. If any one body is traced out, the remaining two will be relatively Gaussian localised. $|LK3\rangle$ therefore has similar properties to $|LK\rangle$; it is a momentum eigenstate, its three bodies are relatively localised and its centre of mass state has no localisation.

With $|LK\rangle$ and $|LK3\rangle$ I can now give a description of the thought experiment of section 1. The experiment begins with a two body universe described by $|LK\rangle$, one body being the original object ready to emit an electron, the other a notional external observer. Once the electron has been emitted the description shifts to $|LK3\rangle$ with object, electron and external observer. The object and the electron are each localised relative to the external observer, so their joint centre of mass must be also. There is no longer anything to prevent this final centre of mass state from being the same as the initial position state of the object alone, so requirement iv can be met. I know of no reason why the 'external observer' should not be the centre of mass of the rest of our universe, in which case the experiment has been brought into the real world. The following is equivalent to a proof that this is indeed possible.

It is clear that the form of $|LK3\rangle$ can be extended to any arbitrary number of particles n.

$$|LKn\rangle = \int_{k_1} \dots \int_{k_{n-1}} e^{-k_1^2/\sigma_k^2} e^{ik_1 x_{1n}} \dots e^{-k_{n-1}^2/\sigma_k^2} e^{ik_2 x_{n-1,n}} \times$$
$$\times |k_1\rangle_1 \otimes \dots \otimes |k_{n-1}\rangle_{n-1} \otimes |-k_1 - \dots - k_{n-1}\rangle_n \, dk_1 \dots dk_{n-1} \qquad (9)$$

$$= \left(\frac{2\sqrt{\pi}}{\sigma_x}\right)^{n-1} \int_{x_1} \dots \int_{x_n} e^{-(x_1 - x_n + x_{1n})^2/\sigma_x^2} \dots e^{-(x_{n-1} - x_n + x_{n-1,n})^2/\sigma_x^2} \times$$
$$\times |x_1\rangle_1 \otimes \dots \otimes |x_n\rangle_n \, dx_1 \dots dx_n \quad (10)$$

A state like this can encompass the whole universe, therefore a universe containing many localised particles can nevertheless be in a momentum eigenstate. Within this universe it seems that the thought experiment, and processes like it, can occur meeting requirements i, ii and iv.

The next step is to make the state an energy eigenstate as well. The resulting state, $|LKE\rangle$, contains a third component representing the internal state of one or more of the bodies. The energy of the internal state is anticorrelated with the kinetic energy of the free bodies, so the total energy of the whole state is a well-defined constant E. Before doing this I want to bring out some mathematical and physical ramifications of that step by trying an analogous thing in a much simpler setting.

3 An Analogy in Spin States

Consider the correlated spin state written in the z-component basis

$$|XZ\rangle = |\uparrow\rangle_1 \otimes |\downarrow\rangle_2 + |\downarrow\rangle_1 \otimes |\uparrow\rangle_2 \qquad (11)$$

Changing to the x-component basis this is

$$|XZ\rangle = |\rightarrow\rangle_1 \otimes |\rightarrow\rangle_2 - |\leftarrow\rangle_1 \otimes |\leftarrow\rangle_2 \qquad (12)$$

That is, $|XZ\rangle$ is anticorrelated in the z-component basis and correlated in the x-component basis, analogously to $|LK\rangle$. I want to explore the problem of creating $|LKE\rangle$, which is $|LK\rangle$ but with the addition of a third component whose energy is anticorrelated with the kinetic energy of the free bodies, leaving the total energy constant. Analogously, I will attempt to put a third spin in correlation with the possible states in the z-component basis, as in

$$|XZE\rangle = |\uparrow\rangle_1 \otimes |\downarrow\rangle_2 \otimes |\downarrow\rangle_3 + |\downarrow\rangle_1 \otimes |\uparrow\rangle_2 \otimes |\uparrow\rangle_3 \qquad (13)$$

If this state works the way $|LKE\rangle$ is expected to, a change to the x-component basis will produce $|XZE\rangle$ something like

$$|\rightarrow\rangle_1 \otimes |\rightarrow\rangle_2 \otimes |s\rangle_3 - |\leftarrow\rangle_1 \otimes |\leftarrow\rangle_2 \otimes |t\rangle_3 \qquad (14)$$

where $|s\rangle_3$ and $|t\rangle_3$ are two states in $|{\leftarrow}\rangle_3$ and $|{\rightarrow}\rangle_3$.

However, the result of changing basis is no such thing.

$$\begin{aligned}
&|XZE\rangle \\
&= \frac{1}{2\sqrt{2}}(|{\rightarrow}\rangle_1 + |{\leftarrow}\rangle_1) \otimes (|{\rightarrow}\rangle_2 - |{\leftarrow}\rangle_2) \otimes (|{\rightarrow}\rangle_3 - |{\leftarrow}\rangle_3) + \\
&\quad (|{\rightarrow}\rangle_1 - |{\leftarrow}\rangle_1) \otimes (|{\rightarrow}\rangle_2 + |{\leftarrow}\rangle_2) \otimes (|{\rightarrow}\rangle_3 + |{\leftarrow}\rangle_3) \\
&= \frac{1}{\sqrt{2}}((|{\rightarrow}\rangle_1 \otimes |{\rightarrow}\rangle_2 \otimes |{\rightarrow}\rangle_3) - (|{\leftarrow}\rangle_1 \otimes |{\leftarrow}\rangle_2 \otimes |{\rightarrow}\rangle_3) + \\
&\quad (|{\rightarrow}\rangle_1 \otimes |{\leftarrow}\rangle_2 \otimes |{\leftarrow}\rangle_3) - (|{\leftarrow}\rangle_1 \otimes |{\rightarrow}\rangle_2 \otimes |{\leftarrow}\rangle_3)) \qquad (15)
\end{aligned}$$

Which is not the required form. In particular, 1 and 2 are uncorrelated.

Mathematically, this comes about because the final form of $|XZ\rangle$ in the x-component basis is produced by cancellation of x terms arising from different terms in the z-component representation. In $|XZE\rangle$ the z terms of $|XZ\rangle$ are multiplied by new factors that differ, so it is not to be expected that the required cancellation will occur. Exploration with alternative forms for the "energy" states suggests that there is no choice that produces the required result. This does not augur well for the next calculation: $|LK\rangle$ surely has the equivalent property, that terms of different energy contribute to cancellations when changing basis. Nevertheless, there are many important disanalogies between the spin and position-momentum cases, and this particular result does not carry over.

There is an important point that will carry over, however. Suppose for a moment that (14) were the correct form for $|XZE\rangle$ in the x-component basis. Now consider what happens if the z-component of spin 3 is measured. Whichever result appears the anticorrelated state is broken up and the positive correlation in the x-component basis is destroyed. For instance, assume the result is $|{\downarrow}\rangle_3$. Then the new state $|M\rangle$ will be

$$\begin{aligned}
|M\rangle &= |{\uparrow}\rangle_1 \otimes |{\downarrow}\rangle_2 \otimes |{\downarrow}\rangle_3 \\
&= \frac{1}{2\sqrt{2}}(|{\rightarrow}\rangle_1 + |{\leftarrow}\rangle_1) \otimes (|{\rightarrow}\rangle_2 - |{\leftarrow}\rangle_2) \otimes ((|{\rightarrow}\rangle_3 - |{\leftarrow}\rangle_3) \\
&= \frac{1}{2\sqrt{2}}(|{\rightarrow}\rangle_1 \otimes |{\rightarrow}\rangle_2 \otimes |{\rightarrow}\rangle_3) - (|{\rightarrow}\rangle_1 \otimes |{\rightarrow}\rangle_2 \otimes |{\leftarrow}\rangle_3) - \\
&\quad (|{\rightarrow}\rangle_1 \otimes |{\leftarrow}\rangle_2 \otimes |{\rightarrow}\rangle_3) + (|{\rightarrow}\rangle_1 \otimes |{\leftarrow}\rangle_2 \otimes |{\leftarrow}\rangle_3) + \\
&\quad (|{\leftarrow}\rangle_1 \otimes |{\rightarrow}\rangle_2 \otimes |{\rightarrow}\rangle_3) - (|{\leftarrow}\rangle_1 \otimes |{\rightarrow}\rangle_2 \otimes |{\leftarrow}\rangle_3) - \\
&\quad (|{\leftarrow}\rangle_1 \otimes |{\leftarrow}\rangle_2 \otimes |{\rightarrow}\rangle_3) + (|{\leftarrow}\rangle_1 \otimes |{\leftarrow}\rangle_2 \otimes |{\leftarrow}\rangle_3) \qquad (16)
\end{aligned}$$

Which is easily distinguished from (14) by measurements made on spins 1 and 2 alone. In (14) only anticorrelated results are possible, while in $|M\rangle$ the other combinations are equally likely. Hence, if $|XZE\rangle$ really had the form (14) in the x-component basis it would provide the core of a superluminal communicator.

An experimenter with access to 1 and 2 could tell whether a second experimenter at a different place had measured 3. As I have shown, $|XZE\rangle$ cannot be used as a communicator because its correct form is (15) rather than (14). In the next two sections it will become clear that $|LKE\rangle$ does have a form which allows it in principle to be used as a superluminal communicator.

4 Energy, Momentum and Localisation

In this section I will construct the state $|LKE\rangle$, which is an extension of $|LK\rangle$ to an energy eigenstate. I will construct an energy-momentum eigenstate in the momentum basis, and then show that it is a position localised state by transforming to the position basis. In this case, the integrals involved are much more difficult, and it will be necessary to adopt a set of simplifying approximations, both in the form of the states carrying the residual energy and the distribution over k. These approximations should not have any substantial effect on the physics of the final result, but even with their help, it will be necessary to evaluate the integrals by numerical means.

The first question to be addressed is the form of the states that will carry the residual energy. In the original electron emission example the residual energy would remain in the internal states of the electron gun itself, somewhere in the conduction or valence bands of the metal. I am looking for a proof in principle only, so I want to avoid the mathematical difficulty of a full model for the internal states. I will adopt the simplest possible approximation having the necessary properties. These are:

a. continuous energy eigenvalues,

b. zero momentum.

The internal energy states of solids have both these properties. Even though confined similarly to a particle in a box, the valence and conduction electrons of a solid have available continuous bands of energy levels. And having zero total momentum is a defining property for an internal state: the total momentum of an isolated body appears in its centre of mass state; the internal momentum must be zero.

The simplest continuum energy state is an infinite plane standing wave, but this is not a zero momentum state, although its mean momentum is zero. To get zero momentum we will need to consider a complete two-particle bound state. The relationship between the two particles of a bound state is very similar to that of the two free particles in $|LK\rangle$ itself: at any instant their momenta are equal and opposite. I will make the drastic simplifications of not considering the force binding the two particles together, and of not confining them to a finite volume. The model this leaves is of two particles in a

momentum-anticorrelated infinite standing wave. The energy state will be

$$\begin{aligned}|E'\rangle &= |k'\rangle_1 \otimes |-k'\rangle_2 + |-k'\rangle_1 \otimes |k'\rangle_2 \\ &= \int_{y_1}\int_{y_2}\left(e^{ik'(y_1-y_2)} + e^{-ik'(y_1-y_2)}\right)|y_1\rangle_1 \otimes |y_2\rangle_2\, dy_1 dy_2 \\ &= \int_{y_1}\int_{y_2} \cos k'(y_1-y_2)\,|y_1\rangle_1 \otimes |y_2\rangle_2\, dy_1 dy_2 \qquad (17)\end{aligned}$$

Where y_1 and y_2 are the position variables for the two particles and $k' = \sqrt{E'}$ (With mass set to 1). (Recall that $\hbar = c = 1$.) The residual energy for a given value of k in $|LK\rangle$ is $E - k^2$, where E is the total energy (and the masses of the free bodies have also been set to 1). Hence, the internal energy state for a given value of k is

$$\left|E - k^2\right\rangle = \int_{y_1}\int_{y_2} \cos\left((y_1 - y_2)\sqrt{E-k^2}\right)|y_1\rangle_1 \otimes |y_2\rangle_2\, dy_1 dy_2 \qquad (18)$$

The next question that must be decided is the distribution over k for $|LKE\rangle$. This is a problem because energy can only be positive. A large value of kinetic energy in the free bodies cannot be balanced by a negative value of the residual energy. There must be an upper bound to the energy, and hence an upper bound to the absolute value of k. The combination of the energy factor and the finite range of k means that the change-of-basis integral for $|LKE\rangle$ will be much more difficult to solve than that of $|LK\rangle$. To keep the problem as simple as possible I will choose a square distribution for k. The final form of $|LKE\rangle$ is therefore

$$\begin{aligned}|LKE\rangle = \int_{k=-k_0}^{k_0} e^{ikx_{12}}\,|k\rangle_1 \otimes |-k\rangle_2 \otimes \Big(\left|\sqrt{E-k^2}\right\rangle_3 \otimes \left|-\sqrt{E-k^2}\right\rangle_4 + \\ + \left|-\sqrt{E-k^2}\right\rangle_3 \otimes \left|\sqrt{E-k^2}\right\rangle_4\Big)\, dk \qquad (19)\end{aligned}$$

where $E \geq k_0^2$ is the total energy of the state. Now transform to the position basis.

$$\begin{aligned}|LKE\rangle = \int_{x_1}\int_{x_2}\int_{y_3}\int_{y_4}\int_{k=-k_0}^{k_0} e^{ik(x_1-x_2+x_{12})}\cos(\sqrt{E-k^2}\,(y_3-y_4))dk \times \\ \times |x_1\rangle_1 \otimes |x_2\rangle_2 \otimes |y_3\rangle_3 \otimes |y_4\rangle_4\, dx_1 dx_2 dy_3 dy_4 \qquad (20)\end{aligned}$$

with $E \geq k_0^2$.

The integral in k can be simplified in two ways. First, by substituting $x = x_1 - x_2 + x_{12}$ and $y = y_3 - y_4$. Second, by substituting $e^{it} = \cos t + i \sin t$ and noting that the imaginary part is antisymmetric and therefore its integral,

over symmetric limits, is zero. This leaves

$$\Psi(x,y) = \int_{k=-k_0}^{k_0} \cos(xk)\cos\left(y\sqrt{E-k^2}\right) dk \tag{21}$$

as the integral that must be solved. This form has two close relatives in Gradshteyn ([3] 3.711 p439, 3.876-7 p508), but, despite considerable effort on the problem, I was forced to resort to numerical analysis to find the form of its solution. The calculation was performed using the quadrature routines of the Nag Fortran Library version 16; specifically, routine D01AKF, running on a Sun SPARCstation IPC.

Figure 1 shows the result of integral (21) evaluated for $k_0 = 1$, $E = 1$, over a range of -25 to 25 for both x and y, at a spacing of 1. This plot was generated using Mathematica 2.2, as were those following. It is clear that the state is localised in x, though not as sharply as it would be if the y subsystem did not exist. In fact, for $y = 0$ the integral is exactly what it would be if the residual energy states did not exist. However, for values of x away from zero the maximum amplitude is not at $y = 0$ but at a value near to $y = |x|$. This means that values of x away from zero are more probable overall than they would be without the residual energy component, and so the localisation is less sharp. Nevertheless, barring unexpected behaviour outside the region so far explored, which is very unlikely given the good behaviour of the two related integrals above, there is little doubt that the system is localised in x.

The really important issue here is the relative localisation of x_1 and x_2, irrespective of the behaviour of y_3 and y_4. To display the x-localisation more clearly I trace out the internal states, which amounts to squaring the result of the first integration and then integrating over y for each value of x. This has the effect of averaging over y. Values were calculated over the range -40 to 40, at a spacing of .25, for x and y, then the integration over y was performed using Nag routine D01GAF. The resulting unnormalised position distribution $d(x)$ is displayed in figure 2. The system is localised in x, and hence is relatively localised in x_1, x_2 with mean separation x_{12}.

The state $|LKE\rangle$ fulfils the requirements i - iv set out in section one. Equation (19) clearly shows it is an energy-momentum eigenstate, while figure 2 shows that it is well localised. Remembering that our expectations of requirement iv have been altered by interpreting $|LKE\rangle$ as describing an entire two-body universe, the state has all the properties demanded of it. $|LKE\rangle$ is a position-localised, energy-momentum eigenstate.

The numerical nature of the above calculation makes it impossible to generalise directly to arbitrary numbers of particles, as needs to be done to sustain the interpretation of $|LKE\rangle$ as encompassing the entire universe. The fact that a state like $|LKE\rangle$ exists at all is telling, and given also that $|LK\rangle$ generalises to $|LKn\rangle$, there is every reason to expect that $|LKE\rangle$ will do likewise.

As a check, I will do the calculation for the next case, $|LKE3\rangle$. The state is

$$|LKE3\rangle = \int_{k_1=-k_0}^{k_0} \int_{k_2=-k_0}^{k_0} e^{ik_1 x_{13}} e^{ik_2 x_{23}} \left|k_1\right\rangle_1 \otimes \left|k_2\right\rangle_2 \otimes \left|-k_1-k_2\right\rangle_3 \otimes \\ \otimes \left(\left|\sqrt{E-E_k}\right\rangle_4 \otimes \left|-\sqrt{E-E_k}\right\rangle_5 + \\ + \left|-\sqrt{E-E_k}\right\rangle_4 \otimes \left|\sqrt{E-E_k}\right\rangle_5 \right) dk_1 dk_2 \quad (22)$$

with $E \geq 3k_0^2$ and $E_k = (1/2)(k_1^2 + k_2^2 + (k_1+k_2)^2)$ (Again with all $m = 1$). Change of basis:

$$= \int_{x_1} \int_{x_2} \int_{x_3} \int_{y_4} \int_{y_5} \int_{k_1=-k_0}^{k_0} \int_{k_2=-k_0}^{k_0} e^{i(k_1(x_1-x_3+x_{13})+k_2(x_2-x_3+x_{23}))} \times \\ \times \cos\left((y_4-y_5)\sqrt{E - \frac{1}{2}({k_1}^2 + {k_2}^2 + (k_1+k_2)^2)} \right) dk_1 dk_2 \times \\ \times \left|x_1\right\rangle_1 \otimes \left|x_2\right\rangle_2 \otimes \left|x_3\right\rangle_3 \otimes \left|y_4\right\rangle_4 \otimes \left|y_5\right\rangle_5 dx_1 dx_2 dx_3 dy_4 dy_5 \quad (23)$$

with $E \geq 3k_0^2$. To solve the integral in k_1, k_2 make equivalent simplifications to those for $|LKE\rangle$. Substitute $xx_1 = x_1 - x_3 + x_{13}$, $xx_2 = x_2 - x_3 + x_{23}$ and $y = y_1 - y_2$; and substitute $e^{it} = \cos t + i \sin t$. Again, the imaginary part is zero because of its integrand's symmetry properties. This can be seen by dividing the integration range into quadrants:

$$\int_{k_1=-k_0}^{k_0} \int_{k_2=-k_0}^{k_0} \ldots = \int_{k_1=0}^{k_0} \int_{k_2=0}^{k_0} \ldots + \int_{k_1=-k_0}^{0} \int_{k_2=-k_0}^{0} \ldots + \\ + \int_{k_1=0}^{k_0} \int_{k_2=-k_0}^{0} \ldots + \int_{k_1=-k_0}^{0} \int_{k_2=0}^{k_0} \ldots \quad (24)$$

The first two of these terms cancel, as do the third and fourth.

This integral was solved using Nag routine D01DAF for two dimensional integrals over a product region, using the values $k_0 = 1$, $E = 3$. Solutions were generated for $0 \leq xx_1 \leq 40$, $-20 \leq xx_2 \leq 20$ at intervals of 1. This is sufficient since the integral is clearly symmetric on reflection in the lines $xx_1 = |xx_2|$. The range of y was $-40 \leq y \leq 40$ at intervals of .25. The internal energy states were traced out by integrating over y the square of the result for each value of xx_1 and xx_2, using routine D01GAF as before. The resulting unnormalised position distribution $d(xx_1, xx_2)$ is displayed in figure 3.

Figure 3 shows a state localised in $xx_1 = x_1 - x_3 + x_{13}$ and $xx_2 = x_2 - x_3 + x_{23}$, and hence having relative localisation in x_1, x_2 and x_3. The three particle state has the required properties, it is an energy-momentum eigenstate with all three particles localised relative to each other.

I consider the expectation that $|LKE\rangle$ will generalise to $|LKEn\rangle$ for arbitrary values of n to be a good one. For the purposes of the remaining argument I will assume that $|LKEn\rangle$ exists and has equivalent properties to $|LKE\rangle$ and $|LKE3\rangle$, while admitting that a proof of this has not been given.

5 Nonlocality

Section three was an investigation of the possibilities of adding a third correlate into a two-particle correlated state. As such it was not very informative. But the discussion there raised the possibility that three-component correlated states might be nonlocal, and nonlocal in a way allowing transmission of information. Since it turns out that $|LKE\rangle$ is a three-component correlated state analogous to the one envisioned in section three, I will look in it for the expected nonlocality.

The nonlocality is easy to demonstrate. As figure 2 shows, $|LKE\rangle$ is well localised in position. It is clear from the momentum representation of $|LKE\rangle$ that the residual energy eigenstates are in two to one correspondence with the free particle momentum eigenstates for bodies 1 and 2 (the two momentum states are identical with k_1 and k_2 interchanged). If the residual energy were to be measured exactly the free bodies would be left in a superposition of these two momentum eigenstates. The momentum eigenstates are completely delocalised, and this change would be readily detectable to any observer.

If the residual energy were all in the internal state of body one, an observer on body one could signal an observer on body two by measuring body one's internal energy. This is a fairly drastic kind of signal – complete delocalisation of the universe – but a signal is a signal. Even if the residual energy were divided between the two bodies' internal states, a measurement of one internal energy would narrow the distribution of the relative momentum, leading to a broadening of the relative position distribution. This effect is in principle observable even if quite small.

For $|LKEn\rangle$ the nonlocality is harder to see, but can be shown to exist for any value of n, though its exact form is more difficult to find. The following analysis does not capture the full extent of the nonlocality, but it is sufficient to prove the point. Take $|LKE3\rangle$, equation (23), as an example. If we measure the residual energy and get the result E_m, the state will change in several ways. First, the residual energy factor is now independent of k_1 and k_2 (it becomes $\cos((y_4 - y_5)\sqrt{E_m})$), and so drops out of the k-integral. At the same time, k_1 and k_2 cease to be independent of one another; they are related by

$$\frac{1}{2}\left(k_1^2 + k_2^2 + (k_1 + k_2)^2\right) = E - E_m \tag{25}$$

with $0 \leq E_m \leq E$. The restriction is demanded by conservation of energy (recall that E has been chosen to equal the maximum possible kinetic energy $3k_0^2$, if E were larger the restriction would need to be suitably modified). The effect of

this is to make the distribution of the free body momenta narrower, necessarily broadening the distribution of the positions of the three bodies. In the extreme case as E_m approaches E we have k_1, k_2, k_3 all approach 0: the positions of all bodies will be completely delocalised as the k-distribution becomes infinitely narrow.

As before, measuring the residual energy produces an observable effect elsewhere in the universe. This effect is chance dependent, since the amount of broadening depends on the value found for E_m, but remains a real nonlocality despite this. Furthermore, it is clear that a similar effect will occur for any number of free bodies: when E_m is close to E all the bodies will be substantially delocalised because of the narrow distribution allowed for their kinetic energies. Provided $|LKEn\rangle$ has the expected properties for arbitrary n, a detectable nonlocality can be manifested in a universe of arbitrary size, clearly including one the size of ours.

There remains one questionmark over this nonlocality. The supposedly nonlocal behaviour occurs only after a measurement has been performed on every object in the universe to find the total residual energy. It could be argued that this phenomenon is not a true nonlocality: that although the behaviour looks superficially like nonlocality it is not capable of carrying signals. That would not be surprising; it is the most widely accepted interpretation of the Bell and GHZ nonlocality theorems for spin states (see e.g. d'Espagnat[4] and Greenberger et.al.[5]). The states cannot be used to transmit signals even though they are nonlocal in a formally defined sense.[2]

This argument is incorrect. It depends on the assumption that the residual energy is spread through the internal states of all the bodies in the universe. While this is most likely true for our universe, it is in principle possible for all the residual energy to be held within a confined volume, and then the delocalisation of the bodies within the universe after its measurement would be nonlocal beyond doubt.

The important point of this section is the apparent existence of a nonlocality allowing transmission of signals which is inherent in states of the form $|LKE\rangle$, and therefore inherent in quantum mechanics.

6 Conclusion

This paper has described a new form of quantum state, $|LKE\rangle$, with two important properties. First, it is a localised energy-momentum eigenstate. Second, it has the potential to exhibit nonlocal behaviour in the positive sense of sending signals faster than light. These remarkable qualities are offset by an awkward facet: $|LKE\rangle$ is not suitable in general for describing ordinary isolated objects in our own universe because the centre of mass position of objects so described

[2] Although it should be remembered that a case for an underlying physical nonlocality could be made, at least within a realist framework.[6, 7]

cannot be localised. The most natural use for these states is to describe an entire universe, encompassing every body contained in it. Here the lack of localisation of the centre of mass becomes a virtue, reflecting the non-absoluteness of position. The need to encompass an entire universe makes $|LKEn\rangle$ difficult to use, because calculations for it can so far only be done numerically and no proof exists that $|LKEn\rangle$ has the correct properties for all n, although a failure of this would seem unlikely. The difficulty in analysing $|LKEn\rangle$ mathematically also explains why I have only offered proofs in principle for the existence of these states, rather than a more physically realistic treatment.

Given the peculiar properties of states like $|LKE\rangle$, why should we imagine that our universe is in one? I have no conclusive arguments to offer for this, but I can offer some plausibility arguments. First of all, the idea must have a strong intuitive appeal. Our expectations about energy-momentum conservation are most simply and directly realised if the universe is in an energy-momentum eigenstate. In the case of momentum, it is not clear that non-zero momentum for the centre of mass of the universe is a possibility we can give meaning to. It certainly is not consistent with any notion of momentum as a relative quantity. Only if the centre of mass state of the universe is a momentum eigenstate of momentum zero is this problem definitely dealt with.

The equivalent argument does not hold for energy, since it is not a relative quantity. However, there is little doubt that the total energy of the universe has consequences detectable from within; certainly in the big bang model the total energy affects the history and development of the universe in an observable way. If the universe is not an energy eigenstate, then its observable state may not be well defined.

A more subtle argument is that $|LKEn\rangle$ would fill a hole in our understanding of energy conservation in quantum mechanics. We have the previously mentioned (see footnote 1) theorem that energy is conserved, in the sense that its mean value and distribution are conserved, by Hamiltonian time evolution under a time-independent Hamiltonian operator. Yet we are in a peculiar position in that many of the detail processes we know of, such as photon emission (e.g. Davies[8] p108), have not been shown to conserve energy explicitly. The universe model suggested by $|LKEn\rangle$ is one in which we know why individual processes conserve energy, and therefore why the universe as a whole does. If the big bang theory of cosmology is correct, then the intimate entanglement of all the bodies in the universe is explained by their component particles being able to trace, directly or indirectly, a history back to an epoch when all the contents of the universe were in close interaction.

None of these arguments is conclusive, but one important point remains even if they are discounted. In the past the possibility of a universal energy-momentum eigenstate was not considered because there was no realisation that such a thing was possible. Now it seems the possibility exists, and it deserves investigation.

Leading naturally to the most important question, that of experi-

mental tests. The first potential test of whether the universe is in a state like $|LKEn\rangle$ is that localisation of the bodies is broader than it would be for an isolated body having the same momentum distribution. The isolated body's wave function is the same as the $y = 0$ slice on the plot, which clearly has smaller amplitude for large x than that for $y \approx x$. It might be possible to perform an experiment to detect this broadening of position uncertainty. Remember, however, that the k-distribution of $|LKE\rangle$ was chosen only for convenience of calculation. It is certainly not the best choice, and there is no telling how other distributions might affect the discrepancy between the isolated body and $|LKE\rangle$ position uncertainties. Tests might also arise from the nonlocal properties of $|LKE\rangle$, or from cosmological consequences arising from the state. I cannot speculate about the form such tests might take.

We have been handed a surprise in $|LKE\rangle$ for a reason I have not so far discussed. When explaining the counterintuitive properties of quantum mechanics one of the favourite examples has been the electron two-slit interference experiment (Feynman,[9] p1-5). A point that is always emphasised is the fact that the interference pattern vanishes if we attempt to determine which slit each electron passed through. This principle is generalisable to the idea that whenever interference occurs between multiple channels, or pathways, the interference exists only so long as it remains impossible to determine which channel the system passed through. The moment a correlation is established that would allow us to determine which channel the system occupied the interference effect vanishes. This behaviour shows up in many places, such as two particle double-slit experiments.[10] I have not been able to find a general proof of this principle; it seems to be widely accepted on the lack of counterexamples.

The $|LKE\rangle$ states form a counterexample. The localisation peak is an interference maximum formed by a sum over many momentum states, each of which forms a separate channel. Yet the momentum states are correlated with the residual energy states, so that measuring the residual energy determines which interference channel the free particle occupies. Apparently, it is possible to arrange a state with interference channels in correlation with the measurable state of some other system in certain circumstances. Indeed, it is this very fact that opens the possibility of superluminal signalling.

The nonlocal character of this class of states has a more immediate impact on quantum theory than its other properties. Whether the universe is in such a state or not, the existence of $|LKE\rangle$ is a proof in principle that quantum mechanics is nonlocal, and demands either a general proof that such states cannot exist in our universe, or a reappraisal of the relationship between quantum and relativity theory.

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Figure 1: 2-body universe wavefunction

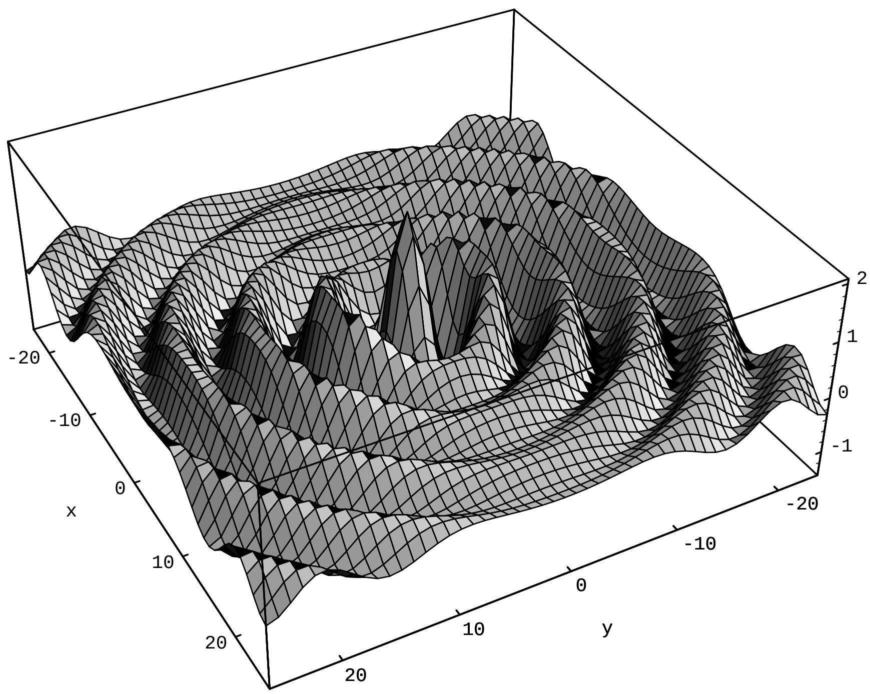

Figure 2: 2-body universe position distribution

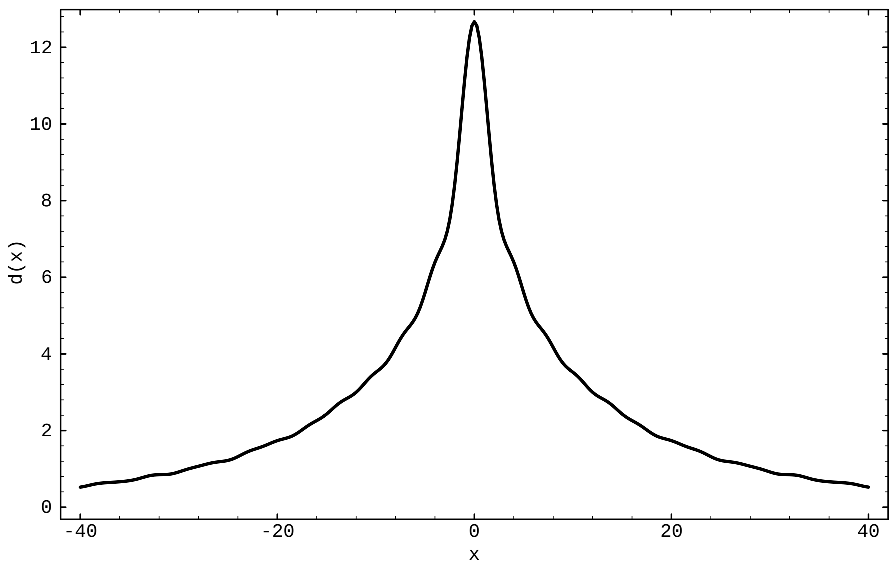

Figure 3: 3-body universe position distribution